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A simple and accurate solution for calculating stresses in conical shells

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Abstract

In this paper, new variable transformation formulas are introduced to solve the basic governing differential equations for conical shells. By performing magnitude order analysis and neglecting the quantities with h/R magnitude order, the basic governing differential equations for conical shells are transformed into a second-order differential equation with complex constant coefficients. By solving this second-order differential equation, a simple and accurate solution for conical shells is derived. The present solution is simpler than the exact solution because it does not use Bessel's functions, and also more accurate than the equivalent cylinder solution. Numerical examples are given to illustrate this conclusion. The simple and accurate solution provides a quick means for analyzing stresses in conical shells. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Conical shell; Stress analysis; Exact solution; Approximate solution; Simple and accurate solution

1. Introduction

Axisymmetric pressure hulls, such as cone and cylinder shells are important configurations which have been widely used in engineering structures, such as submarine and submersible pressure hulls [1]. Calculation of stresses in these structures is a necessity in design. For most of the axisymmetric thin-walled shells, exact solutions can be found but they are very complex except for cylinders [2,3]. Special functions, such as Bessel functions, Thomson functions and Hankal functions which are not very familiar to many engineers need to be used. Therefore, these exact solutions have not found wide applications in practice. Finite element methods have been recommended in Ref. [1] but in terms of efficiency, this may not be so favorable, especially in preliminary design. Furthermore, in comparison with the analytical solutions, finite element methods provide less insight in understanding the fundamental mechanical behaviour of the structural parameters. Therefore, analytical solutions are still sought by some researchers. For the general axisymmetric structures, there are two different approaches in this aspect. One is to use the concept of equivalent cylindrical shell, that is, to define an equivalent cylinder for the general axisymmetric shell and then apply the cylinder solution. This has been used in Ref. [4] for submarine pressure hull design. However, the accuracy of these solutions has been found to be poorer than the exact solutions. Therefore, these are called approximate solutions in this paper. The other is to introduce some new types of variable transformations and apply the magnitude order analysis to simplify the governing differential equation, simple and accurate solutions can also be derived.

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Nomenclature

α	half vertex angle of the conical shell
φ	complement of the half vertex angle
h	thickness of the conical shell
E	Young's modulus of elasticity
μ	Poisson's ratio
D	bending stiffness
S	coordinate along the generator of the conical shell
R_2	principal radius of curvature of the conical shell
w	displacement in the radial direction
N_1	transverse shear force
T_1	stress resultant in the generator direction
T_2	stress resultant in radial direction
T_1^*	membrane stress resultant in the generator direction
T_2^*	membrane stress resultant in radial direction
M_1	bending moment in the generator direction
M_2	bending moment in radial direction
Δ_x	displacement along the circumferential direction
σ_1	stress in the generator direction
σ_2	stress in circumferential direction
q_1	face load per unit area along the generator direction
q_n	face load per unit area along the radial direction
q	dead weight per unit area
P	concentrated load
l	slant length of the conical shell
r	radius of parallel circular of the conical shell

In Ref. [3], such a simple and accurate solution for the general shell of revolution was presented based on the variable transformation formulas they introduced. The solution was expressed only by elementary functions such as exponential functions and triangular functions. Conical shells are the shells of revolution, however, when we applied their variable transformation formulas to conical shells, it was found that these formulas are singular. In order to overcome this problem, a new type of variable transformations will be introduced in this paper. By using these new variable transformation formulas together with the basic governing differential equations, a simple and accurate solution which has the same accuracy as the exact solution will be derived. The main purpose of this paper is to present such a solution. In order to compare the present solution with the exact solution and the approximate solution for conical shells, the latter two solutions are briefly introduced in Sections 2 and 3, respectively. In Section 5, a numerical example is used to compare the present solution with both exact solution and the approximate solution.

2. The exact solution for conical shells

Let 2α represent the vertex angle of a conical shell, S represents the coordinate along the generator of the conical shell from the vertex of the conical shell (Fig. 1). For conical shells, the principal radii of curvature are $R_1 = \infty$ and $R_2 = S\tan\alpha$. By substituting these conditions to the governing differential equations for the general shell of revolution [2,3], the following governing differential equations for conical shells can be obtained:

$$\begin{aligned} L(U) &= -Eh\gamma_1, \\ L(\gamma_1) &= \frac{U}{D}, \end{aligned} \tag{1}$$

where

$$U = N_1 R_2, \tag{2}$$

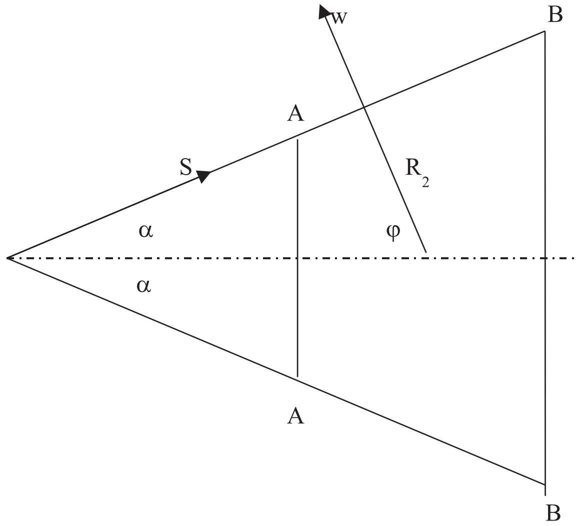


Fig. 1. A conical shell with parameters defined.

$$\gamma_1 = \frac{\mathrm{d}w}{\mathrm{d}S},$$
$$L(\cdots) = \tan\alpha\left[S\frac{\mathrm{d}^2(\cdots)}{\mathrm{d}S^2} + \frac{\mathrm{d}(\cdots)}{\mathrm{d}S} - \frac{(\cdots)}{S}\right], \tag{3}$$

where $L(\cdots)$ is the Laplace operator; N_1, the transverse shear force; w, the displacement in the radial direction; h, the thickness of the conical shell, E, Young's modulus of elasticity, and D, the bending stiffness which is

$$D = \frac{Eh^3}{12(1-\mu^2)}, \tag{4}$$

where μ is Poisson's ratio. Substituting the first formula of Eq. (1) into the second formula of Eq. (1), we have

$$LL(U) + \lambda^4 U = 0, \tag{5}$$

where

$$\lambda^4 = \frac{Eh}{D}, \tag{6}$$

Eq. (5) can be split into two conjugate differential equations of second order:

$$L(U) \pm \mathrm{i}\lambda^2 U = 0. \tag{7}$$

Using the second formula of Eq. (3) and $U = N_1 R_2$, $R_2 = S\tan\alpha$, Eq. (7) can be written as

$$\frac{\mathrm{d}^2(N_1S)}{\mathrm{d}S^2} + \frac{1}{S}\frac{\mathrm{d}(N_1S)}{\mathrm{d}S} + \left(-\frac{1}{S^2} \pm \frac{\mathrm{i}\lambda^2}{S\tan\alpha}\right)(N_1S) = 0. \tag{8}$$

It is a pair of conjugate differential equations of second order. The solutions to Eq. (8) are also a pair of conjugate complex functions. As long as the solution to any one of the Eq. (8) is obtained, then the real part and the imaginary part of the solution are derived. Using these two parts, we can find the general solution for differential equation (5).

Introducing the following transformations

$$\eta = x\sqrt{\mathrm{i}} = 2\lambda\sqrt{\frac{1}{\tan\alpha}}\sqrt{S}\sqrt{\mathrm{i}},$$
$$x = 2\sqrt[4]{3(1-\mu^2)}\sqrt{\frac{2}{h\tan\alpha}}\sqrt{S}, \tag{9}$$

and using any one of two equations in Eq. (8), we have

$$\frac{\mathrm{d}^2(N_1S)}{\mathrm{d}\eta^2}+\frac{1}{\eta}\frac{\mathrm{d}(N_1S)}{\mathrm{d}\eta}+\left(1-\frac{4}{\eta^2}\right)(N_1S)=0 \tag{10}$$

which is a Bessel equation. The solution to this equation is

$$N_1S=C_1J_2(\eta)+C_2H_2^{(1)}(\eta). \tag{11}$$

In Eq. (11), $J_2(\eta)$ is the first type of Bessel functions of second order; $H_2^{(1)}(\eta)$, the first type of Hankal functions of second order, and C_1, C_2 are complex constant coefficients. The functions mentioned above can be expressed by means of Thomson functions and their first derivatives [5],

$$\begin{aligned} J_2(\eta)&=\left(\frac{2}{x}bei'x-berx\right)+\mathrm{i}\left(\frac{2}{x}ber'x+beix\right),\\ H_2^{(1)}(\eta)&=\frac{2}{\pi}\left(\frac{2}{x}\ker'x+keix\right)-\mathrm{i}\frac{2}{\pi}\left(\frac{2}{x}kei'x-\ker x\right), \end{aligned} \tag{12}$$

where the symbol (′) represents the derivative of the function. If we define

$$\begin{aligned} C_1&=-A_1-\mathrm{i}A_2,\\ C_2&=\frac{\pi}{2}(B_2-\mathrm{i}B_1), \end{aligned} \tag{13}$$

and substitute Eqs. (12) and (13) into Eq. (11) and select the real part, we will have

$$N_1=\frac{1}{S}\left[A_1\left(berx-\frac{2}{x}bei'x\right)+A_2\left(beix+\frac{2}{x}ber'x\right)+B_1\left(\ker x-\frac{2}{x}kei'x\right)+B_2\left(keix+\frac{2}{x}\ker'x\right)\right], \tag{14}$$

where A_1, A_2, B_1, B_2 are unknown constants which can be determined from boundary conditions. In many applications the conical shell is long enough to make it possible to neglect the effect of one edge on another edge. In order to simplify the calculation, we can choose the terms with B_1 and B_2 for edge AA of Fig. 1 and the terms with A_1 and A_2 for edge BB of Fig. 1.

From the definition of $U=N_1R_2$, U can also be obtained from Eq. (14). Using the first formula of Eq. (1) and the second formula of Eq. (3), we can obtain

$$\gamma_1=-\frac{L(U)}{Eh}=-\frac{\tan^2\alpha}{Eh}\left[S\frac{\mathrm{d}^2(N_1S)}{\mathrm{d}S^2}+\frac{\mathrm{d}(N_1S)}{\mathrm{d}S}-N_1\right]. \tag{15}$$

After obtaining U and γ_1, the calculation of the stress resultants T_1 and T_2, bending moments M_1 and M_2, and displacement Δ_x along circumferential direction are performed using the basic definitions [2,3]:

$$\begin{aligned} T_1&=-\frac{U}{R_2}\frac{\cos\varphi}{\sin\varphi}=-N_1\tan\alpha,\\ T_2&=-\frac{\mathrm{d}U}{\mathrm{d}S}=-\frac{\mathrm{d}(N_1S)}{\mathrm{d}S}\tan\alpha,\\ M_1&=D\left(\frac{\mathrm{d}\gamma_1}{\mathrm{d}S}+\frac{\mu\cos\varphi}{R_2\sin\varphi}\gamma_1\right)=D\left(\frac{\mathrm{d}\gamma_1}{\mathrm{d}S}+\frac{\mu}{S}\gamma_1\right),\\ M_2&=D\left(\frac{\cos\varphi}{R_2\sin\varphi}\gamma_1+\mu\frac{\mathrm{d}\gamma_1}{\mathrm{d}S}\right)=D\left(\frac{\gamma_1}{S}+\mu\frac{\mathrm{d}\gamma_1}{\mathrm{d}S}\right),\\ \Delta_x&=\varepsilon_\theta S\sin\alpha=\frac{S\tan\alpha\sin\alpha}{Eh}\left(-\frac{\mathrm{d}(N_1S)}{\mathrm{d}S}+\mu N_1\right). \end{aligned} \tag{16}$$

And the final results for exact solution are expressed as follows:

$$
\begin{aligned}
N_1 = {} & \frac{\sqrt{3(1-\mu^2)}}{h\tan\alpha}\frac{8}{x^2}\left[A_1\left(berx-\frac{2}{x}bei'x\right)+A_2\left(beix+\frac{2}{x}ber'x\right)\right. \\
& \left.+B_1\left(\ker x-\frac{2}{x}kei'x\right)+B_2\left(keix+\frac{2}{x}\ker'x\right)\right], \\
\gamma_1 = {} & -\sqrt{3(1-\mu^2)}\frac{2\tan\alpha}{Eh^2}\left[A_1\left(-beix-\frac{2}{x}ber'x\right)+A_2\left(berx-\frac{2}{x}bei'x\right)\right. \\
& \left.+B_1\left(-keix-\frac{2}{x}\ker'x\right)+B_2\left(\ker x-\frac{2}{x}kei'x\right)\right], \\
T_1 = {} & -\frac{\sqrt{3(1-\mu^2)}}{h}\frac{8}{x^2}\left[A_1\left(berx-\frac{2}{x}bei'x\right)+A_2\left(beix+\frac{2}{x}ber'x\right)\right. \\
& \left.+B_1\left(\ker x-\frac{2}{x}kei'x\right)+B_2\left(keix+\frac{2}{x}\ker'x\right)\right], \\
T_2 = {} & -4\frac{\sqrt{3(1-\mu^2)}}{h}\left[A_1\left(-\frac{2}{x^2}berx+\frac{1}{x}ber'x+\frac{4}{x^3}bei'x\right)+A_2\left(-\frac{2}{x^2}beix-\frac{4}{x^3}ber'x+\frac{1}{x}bei'x\right)\right. \\
& \left.+B_1\left(-\frac{2}{x^2}\ker x+\frac{1}{x}\ker'x+\frac{4}{x^3}kei'x\right)+B_2\left(-\frac{2}{x^2}keix-\frac{4}{x^3}\ker'x+\frac{1}{x}kei'x\right)\right], \\
M_1 = {} & -\left[A_1\left(4(1-\mu)\frac{beix}{x^2}+8(1-\mu)\frac{ber'x}{x^3}-\frac{2}{x}bei'x\right)+A_2\left(-4(1-\mu)\frac{berx}{x^2}+8(1-\mu)\frac{bei'x}{x^3}+\frac{2}{x}ber'x\right)\right. \\
& \left.+B_1\left(4(1-\mu)\frac{keix}{x^2}+8(1-\mu)\frac{\ker'x}{x^3}-\frac{2}{x}kei'x\right)+B_2\left(-4(1-\mu)\frac{\ker x}{x^2}+8(1-\mu)\frac{kei'x}{x^3}+\frac{2}{x}\ker'x\right)\right], \\
M_2 = {} & -\left[A_1\left(-4(1-\mu)\frac{beix}{x^2}-8(1-\mu)\frac{ber'x}{x^3}-\frac{2\mu}{x}bei'x\right)+A_2\left(4(1-\mu)\frac{berx}{x^2}-8(1-\mu)\frac{bei'x}{x^3}+\frac{2\mu}{x}ber'x\right)\right. \\
& \left.+B_1\left(-4(1-\mu)\frac{keix}{x^2}-8(1-\mu)\frac{\ker'x}{x^3}-\frac{2\mu}{x}kei'x\right)+B_2\left(4(1-\mu)\frac{\ker x}{x^2}-8(1-\mu)\frac{kei'x}{x^3}+\frac{2\mu}{x}\ker'x\right)\right], \\
\Delta_x = {} & \frac{\tan\alpha\sin\alpha}{Eh}\left[A_1\left((1+\mu)berx-2(1+\mu)\frac{bei'x}{x}-\frac{x}{2}ber'x\right)+A_2\left((1+\mu)beix+2(1+\mu)\frac{ber'x}{x}-\frac{x}{2}bei'x\right)\right. \\
& \left.+B_1\left((1+\mu)\ker x-2(1+\mu)\frac{kei'x}{x}-\frac{x}{2}\ker'x\right)+B_2\left((1+\mu)keix+2(1+\mu)\frac{\ker'x}{x}-\frac{x}{2}kei'x\right)\right].
\end{aligned} \tag{17}
$$

It is important to select the calculation method of Thomson functions and their derivatives for obtaining stable numerical results. In this paper, Thomson functions and their derivatives are calculated by means of recursive formulas for Thomson functions and their derivatives [5]. Based on the recursive formulas for Thomson functions and their derivatives, it is easy to calculate Thomson functions of any order and their derivatives of any order if we know Thomson functions of zero order and their derivatives of first order. Thomson functions of zero order and their derivatives of first order are calculated using the polynomial approximation formulas given in Ref. [5].

After obtaining the stress resultants T_1 and T_2 and bending moments M_1 and M_2 the stress calculations are performed using the following expressions:

$$
\begin{aligned}
\sigma_1 &= \frac{T_1+T_1^*}{h}\pm\frac{6M_1}{h^2}, \\
\sigma_2 &= \frac{T_2+T_2^*}{h}\pm\frac{6M_2}{h^2},
\end{aligned} \tag{18}
$$

where T_1^*, T_2^* are the stress resultants for membrane solution which are given as follows [2,3]:

$$
\begin{aligned}
T_1^* &= -\frac{1}{S}\left[\int(q_1+q_n\tan\alpha)S\,\mathrm{d}S+C\right], \\
T_2^* &= -q_nS\tan\alpha,
\end{aligned} \tag{19}
$$

where q_1, q_n are the face load per unit area along the S direction and its normal direction, C is a constant to be determined from the membrane boundary condition.

It is necessary to point out that the exact solution for conical shells given above is of an accuracy which is the same as the thin-walled shell theory. Due to the various assumptions made in thin-walled shell theory, the error magnitude order of thin-walled shell theory is h/R, so the error magnitude order of the exact solution for conical shells given here is also h/R.

3. The approximate solution for conical shells

Considering the complexity of the exact solution for the shell of revolution, the approximate solutions based on the concept of an equivalent cylindrical shell are generally used for solving problems on the shell of revolution [3,4]. Let the values of φ, S, r, $R_2 = r/\sin\varphi$ at the large end be φ_0, S_0, r_0, $R_{20} = r_0/\sin\varphi_0$, the cylindrical shell which the thickness is the same as the shell of revolution and the radius is $|R_{20}|$ is defined as the equivalent cylindrical shell. For the equivalent cylindrical shell, the governing differential equation for the axial symmetrical case becomes

$$\frac{\mathrm{d}^4\gamma_1(\text{cylinder})}{\mathrm{d}S^4} + \frac{Eh}{DR_{20}^2}\gamma_1(\text{cylinder}) = 0. \tag{20}$$

The error order between the governing differential equation (20) and the governing differential equation of the general shell of revolution for the axial symmetrical case is $\sqrt{h/R}$. For Eq. (20), the solution can easily be found [2–4]. The solutions for the conical shell as shown in Fig. 1 can be expressed as follows:

$$\begin{aligned}
&Eh\gamma_1 = \frac{\mathrm{d}}{\mathrm{d}S}(Ehw), \qquad U = DR_2\frac{\mathrm{d}^3w}{\mathrm{d}S^3},\\
&T_1 = -D\tan\alpha\frac{\mathrm{d}^3w}{\mathrm{d}S^3}, \qquad T_2 = \frac{1}{R_2}Ehw, \qquad M_1 = D\frac{\mathrm{d}^2w}{\mathrm{d}S^2}, \qquad M_2 = \mu D\frac{\mathrm{d}^2w}{\mathrm{d}S^2},\\
&N_1 = D\frac{\mathrm{d}^3w}{\mathrm{d}S^3}, \qquad Q_x = \frac{D}{\cos\alpha}\frac{\mathrm{d}^3w}{\mathrm{d}S^3}, \qquad \Delta_x = w\cos\alpha,
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
Ehw &= C_1\theta(-\beta S') - C_2\zeta(-\beta S') + C_3\theta(\beta S') + C_4\zeta(\beta S'),\\
\frac{\mathrm{d}}{\mathrm{d}S}(Ehw) &= \beta[C_1\varphi(-\beta S') + C_2\psi(-\beta S') - C_3\varphi(\beta S') + C_4\psi(\beta S')],\\
\frac{\mathrm{d}^2}{\mathrm{d}S^2}(Ehw) &= 2\beta^2[C_1\zeta(-\beta S') + C_2\theta(-\beta S') + C_3\zeta(\beta S') - C_4\theta(\beta S')],\\
\frac{\mathrm{d}^3}{\mathrm{d}S^3}(Ehw) &= 2\beta^3[-C_1\psi(-\beta S') + C_2\varphi(-\beta S') + C_3\psi(\beta S') + C_4\varphi(\beta S')],
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\theta(\beta S') = \mathrm{e}^{-\beta S'}\cos\beta S', \qquad \theta(-\beta S') = \mathrm{e}^{\beta S'}\cos\beta S',\\
&\zeta(\beta S') = \mathrm{e}^{-\beta S'}\sin\beta S', \qquad \zeta(-\beta S') = -\mathrm{e}^{\beta S'}\sin\beta S',\\
&\varphi(\beta S') = \mathrm{e}^{-\beta S'}(\cos\beta S' + \sin\beta S'), \qquad \varphi(-\beta S') = \mathrm{e}^{\beta S'}(\cos\beta S' - \sin\beta S'),\\
&\psi(\beta S') = \mathrm{e}^{-\beta S'}(\cos\beta S' - \sin\beta S'), \qquad \psi(-\beta S') = \mathrm{e}^{\beta S'}(\cos\beta S' + \sin\beta S').
\end{aligned} \tag{23}$$

The unknown constants C_1, C_2, C_3 and C_4 in the expressions given above are determined from boundary conditions. As we know the stress resultants T_1 and T_2 and bending moments M_1 and M_2, the stress calculations are made using Eq. (18). Due to the error introduced in the basic governing differential equation, the accuracy of this solution is of a magnitude of order of $\sqrt{h/R}$.

4. The simple and accurate solution for conical shells

The approximate solution for conical shells is simple but the accuracy is low. The accuracy of the exact solution for conical shells is high but is very complex. Is it possible to derive a solution for conical shells which possesses both accuracy and simplicity? The answer is positive. In Ref. [3], by introducing the following variable transformation formulas:

$$y = \tilde{\gamma}_1 \sqrt{\frac{R_2}{R_1} \sin\varphi}, \tag{24}$$

$$\xi = \sqrt[4]{3(1-\mu^2)} \int_{\varphi_0}^{\varphi} \frac{|R_1|}{\sqrt{|R_2|h}} \mathrm{d}\varphi, \tag{25}$$

$$y_1 = \frac{\sqrt{|R_1|}}{\sqrt[4]{|R_2|c}} y, \tag{26}$$

they are able to derive such a solution for the general shell of revolution, where c is a constant parameter defined in Appendix A. Their solutions are supposed to be valid for the general shell of revolution. However, for conical shells, the principal radius of curvature $R_1 = \infty$, thus the variable transformation formula (24) turns out $y = 0$, the variable transformation formula (25) becomes $R_1 = \infty$ times $\mathrm{d}\varphi = 0$, while the third variable transformation formula, Eq. (26), also becomes $y_1 = \infty\ {}^*0$. Therefore, φ cannot be used as a principal curvature coordinate for conical shells. In order to overcome this singularity, a new set of the variable transformation formulas suitable for conical shells are proposed in this paper. They are

$$y = \sqrt{S}\tilde{\gamma}_1, \tag{27}$$

$$\xi = \frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{h\tan\alpha}} \left(2\sqrt{S} - 2\sqrt{S_0}\right), \tag{28}$$

$$y_1 = \frac{1}{\sqrt[4]{cS\tan\alpha}} y, \tag{29}$$

where c is defined in Eq. (A.18). Following the similar procedure as that used in Ref. [3], the simple and accurate solution has been re-derived for conical shells. The detailed process of derivation is given in Appendix A and the solution is given as follows:

$$\begin{aligned}
\gamma_1 &= \frac{\sqrt{12(1-\mu^2)}}{Eh^2} \frac{\sqrt[4]{R_{20}^5}}{\sqrt[4]{S}} \left[c_1 \mathrm{e}^{-\xi} \cos(\xi + \zeta_1) + c_2 \mathrm{e}^{\xi} \cos(-\xi + \zeta_2)\right], \\
U &= \frac{Eh^2}{\sqrt{12(1-\mu^2)}} \frac{2\lambda^2}{Eh} \sqrt[4]{\frac{R_{20}}{S}} \left[c_1 \mathrm{e}^{-\xi} \sin(\xi + \zeta_1) + c_2 \mathrm{e}^{\xi} \sin(-\xi + \zeta_2)\right] \\
&= \frac{\sqrt[4]{R_{20}^5}}{\sqrt[4]{S}} \left[c_1 \mathrm{e}^{-\xi} \sin(\xi + \zeta_1) + c_2 \mathrm{e}^{\xi} \sin(-\xi + \zeta_2)\right], \\
N_1 &= \frac{\sqrt[4]{R_{20}^5}}{R_2 \sqrt[4]{S}} \left[c_1 \mathrm{e}^{-\xi} \sin(\xi + \zeta_1) + c_2 \mathrm{e}^{\xi} \sin(-\xi + \zeta_2)\right],
\end{aligned} \tag{30}$$

$$\begin{aligned}
T_1 &= -N_1 \tan\alpha \\
&= -\frac{\sqrt[4]{R_{20}^5}}{R_2 \sqrt[4]{S}} \tan\alpha \left[c_1 \mathrm{e}^{-\xi} \sin(\xi + \zeta_1) + c_2 \mathrm{e}^{\xi} \sin(-\xi + \zeta_2)\right], \\
T_2 &= \frac{\sqrt[4]{R_{20}^5}}{\sqrt[4]{S}} \left\{ c_1 \mathrm{e}^{-\xi} \left[-\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}} \cos(\xi + \zeta_1) + \left(\frac{1}{4S} + \frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\right) \sin(\xi + \zeta_1) \right] \right. \\
&\quad \left. + c_2 \mathrm{e}^{\xi} \left[\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}} \cos(-\xi + \zeta_2) + \left(\frac{1}{4S} - \frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\right) \sin(-\xi + \zeta_2) \right] \right\},
\end{aligned} \tag{31}$$

$$
\begin{aligned}
k_1 = {} & \frac{\sqrt{12(1-\mu^2)}}{Eh^2}\frac{\sqrt[4]{R_{20}^5}}{\sqrt[4]{S}}\left\{c_1\mathrm{e}^{-\xi}\left[-\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\sin(\xi+\zeta_1)-\left(\frac{1}{4S}+\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\right)\cos(\xi+\zeta_1)\right]\right. \\
& \left.+c_2\mathrm{e}^{\xi}\left[\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\sin(-\xi+\zeta_2)-\left(\frac{1}{4S}-\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\right)\cos(-\xi+\zeta_2)\right]\right\}, \qquad (32)\\
k_2 = {} & \frac{\sqrt{12(1-\mu^2)}}{Eh^2}\frac{\tan\alpha}{R_2}\frac{\sqrt[4]{R_{20}^5}}{\sqrt[4]{S}}\left[c_1\mathrm{e}^{-\xi}\cos(\xi+\zeta_1)+c_2\mathrm{e}^{\xi}\cos(-\xi+\zeta_2)\right],
\end{aligned}
$$

$$
\begin{aligned}
M_1 = {} & \frac{h}{\sqrt{12(1-\mu^2)}}\frac{\sqrt[4]{R_{20}^5}}{\sqrt[4]{S}}\left\{c_1\mathrm{e}^{-\xi}\left[-\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\sin(\xi+\zeta_1)-\left(\frac{1}{4S}+\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}-\mu\frac{\tan\alpha}{R_2}\right)\cos(\xi+\zeta_1)\right]\right. \\
& \left.+c_2\mathrm{e}^{\xi}\left[\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\sin(-\xi+\zeta_2)-\left(\frac{1}{4S}-\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}-\mu\frac{\tan\alpha}{R_2}\right)\cos(-\xi+\zeta_2)\right]\right\},\\
M_2 = {} & \frac{h}{\sqrt{12(1-\mu^2)}}\frac{\sqrt[4]{R_{20}^5}}{\sqrt[4]{S}}\left\{c_1\mathrm{e}^{-\xi}\left[\left(\frac{\tan\alpha}{R_2}-\frac{\mu}{4S}-\mu\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\right)\cos(\xi+\zeta_1)-\mu\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\sin(\xi+\zeta_1)\right]\right. \\
& \left.+c_2\mathrm{e}^{\xi}\left[\left(\frac{\tan\alpha}{R_2}-\frac{\mu}{4S}+\mu\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\right)\cos(-\xi+\zeta_2)+\mu\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\sin(-\xi+\zeta_2)\right]\right\},
\end{aligned} \qquad (33)
$$

$$
\begin{aligned}
\varepsilon_2 = {} & \frac{1}{Eh}(T_2-\mu T_1)\\
= {} & \frac{1}{Eh}\frac{\sqrt[4]{R_{20}^5}}{\sqrt[4]{S}}\left\{c_1\mathrm{e}^{-\xi}\left[-\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\cos(\xi+\zeta_1)+\left(\frac{1}{4S}+\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}+\mu\frac{\tan\alpha}{R_2}\right)\sin(\xi+\zeta_1)\right]\right. \\
& \left.+c_2\mathrm{e}^{\xi}\left[\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\cos(-\xi+\zeta_2)+\left(\frac{1}{4S}-\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}+\mu\frac{\tan\alpha}{R_2}\right)\sin(-\xi+\zeta_2)\right]\right\}.
\end{aligned} \qquad (34)
$$

The displacement along the circumferential direction Δ_x can be expressed as

$$
\begin{aligned}
\Delta_x = {} & w\cos\alpha+u\sin\alpha=\varepsilon_2R_2\cos\alpha\\
= {} & \frac{R_2\cos\alpha}{Eh}\frac{\sqrt[4]{R_{20}^5}}{\sqrt[4]{S}}\left\{c_1\mathrm{e}^{-\xi}\left[-\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\cos(\xi+\zeta_1)+\left(\frac{1}{4S}+\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}+\mu\frac{\tan\alpha}{R_2}\right)\sin(\xi+\zeta_1)\right]\right. \\
& \left.+c_2\mathrm{e}^{\xi}\left[\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\cos(-\xi+\zeta_2)+\left(\frac{1}{4S}-\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}+\mu\frac{\tan\alpha}{R_2}\right)\sin(-\xi+\zeta_2)\right]\right\}.
\end{aligned} \qquad (35)
$$

5. Numerical examples

In order to test the present solution proposed in this paper, we calculate the stresses of a conical shell shown in Fig. 2 by means of the various solutions given above. The thickness h of the conical shell is 3 cm, the slant length l is 10 cm and

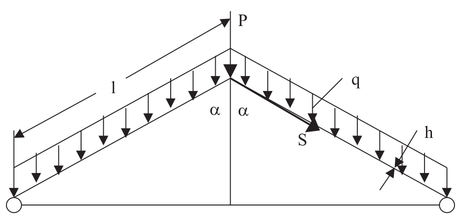


Fig. 2. A closed conical shell under loading.

the half vertex angle α of the conical shell equals 60°. The upper end of the conical shell is closed and the lower edge of conical shell is supported by hinges, in other words, the boundary condition for the lower edge of the conical shell is simply supported boundary condition. Consequently, the circle at the lowest edge is not movable but is able to turn round. In the calculation of stresses two loading cases are considered. Loading case 1 is the dead weight q of the conical shell. Let us assume that the dead weight q per unit area in the middle surface of the conical shell is 1 MPa. Loading case 2 is the concentrated load P. Let P equal 10 000 N. Young's modulus of elasticity E and Poisson's ratio μ are $E = 2.1 \times 10^5$ MPa and $\mu = 0.3$, respectively.

The stress results obtained by all previously given solutions are given in Tables 1 and 2 and Fig. 3. Table 1 makes a comparison between the stress results of the exact solution and of the present solution in loading case 1. It can be observed from Table 1 that the two stress results agree with each other very well. Table 2 gives the same comparison in loading case 2. The comparison shows that the two stress results are still in good agreement except for several small stresses ($<10^{-6}$ MPa). Because of the calculation error it is possible that very small stresses are not precise.

Fig. 3 makes comparison of the stress results obtained by the present solution and the approximate solution in loading case 1. Maximum relative errors are −28.6%, −37.6% and −46.2% for $h = 3.0$, $h = 4.5$ and $h = 6.0$ cm, respectively. It is evident from Fig. 3 that the relative error increases as the thickness of conical shell increases.

6. Conclusions

This paper has presented a simple and accurate solution for conical shells with the same accuracy as the exact solution. The present solution was obtained by introducing a set of new variable transformations and by neglecting the terms with the order of magnitude of h/R. Different from the exact solution which are expressed by special functions, the present solution for conical shells can be expressed only by the elementary functions, such as exponential functions and triangular functions. Numerical examples have been used to compare the stress results obtained by the present solution with that from the exact solution and almost identical results have been obtained. The present solution is also compared with the approximate solution and it is found that quite large error in stresses could be introduced by using the approximate solution for conical shells. Consequently, it can be concluded that the present solution for conical shells is simpler compared to the exact solution and more accurate compared to the approximate solution.

Appendix A

A.1. Derivation of the simple and accurate solution

In the derivation, it is necessary to use the equilibrium equations, geometrical relations and stress–strain relations, so these equations are first given as follows:

$$U = N_1 R_2 = N_1 S \tan\alpha, \tag{A.1}$$

$$\begin{cases} T_1 = -\frac{U}{R_2}\tan\alpha, \\ T_2 = -\frac{\mathrm{d}U}{\mathrm{d}S}, \\ \varepsilon_1 = -\frac{1}{Eh}\left[\frac{U}{R_2}\tan\alpha - \mu\frac{\mathrm{d}U}{\mathrm{d}S}\right], \\ \varepsilon_2 = -\frac{1}{Eh}\left[\frac{\mathrm{d}U}{\mathrm{d}S} - \frac{\mu U}{R_2}\tan\alpha\right], \end{cases} \tag{A.2}$$

$$\begin{cases} k_1 = \frac{\mathrm{d}\gamma_1}{\mathrm{d}S}, \\ k_2 = \frac{\tan\alpha}{R_2}\gamma_1, \\ M_1 = D(k_1 + \mu k_2) = D\left(\frac{\mathrm{d}\gamma_1}{\mathrm{d}S} + \mu\frac{\tan\alpha}{R_2}\gamma_1\right), \\ M_2 = D(k_2 + \mu k_1) = D\left(\frac{\tan\alpha}{R_2}\gamma_1 + \mu\frac{\mathrm{d}\gamma_1}{\mathrm{d}S}\right). \end{cases} \tag{A.3}$$

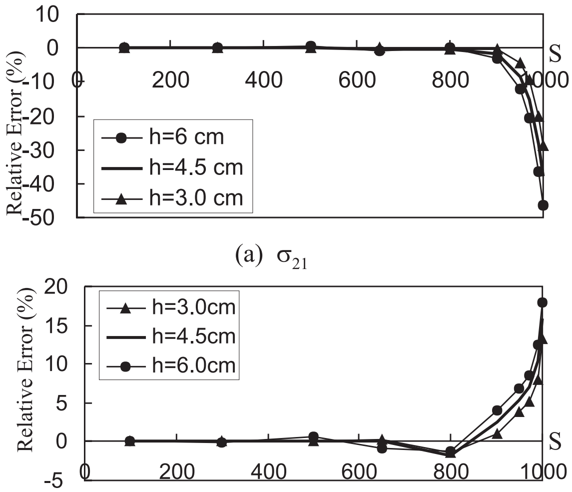
The governing differential equations for conical shells in the coordinate system as shown in Fig. 1 have been given in Eqs. (1)–(3).

Table 1
Comparison of the stress results obtained by the exact solution and the present solution in loading case 1

S (cm)		1000	990	970	950	900	800	650	500	300	100
σ_{11} (MPa)	Exact	−3219.169	−2116.022	−948.6237	−734.0814	−1787.297	−2784.794	−2160.730	−1666.967	−1000.001	−333.333
	Present	−3219.257	−2115.188	−946.3527	−731.2715	−1786.076	−2785.635	−2160.645	−1666.972	−1000.001	−333.333
σ_{12} (MPa)	Exact	−3219.169	−4327.920	−5469.961	−5615.190	−4263.508	−2500.176	−2172.512	−1666.361	−999.9994	−333.333
	Present	−3219.257	−4328.936	−5472.434	−5618.211	−4264.866	−2549.302	−2172.602	−1666.356	−999.9993	−333.333
σ_{21} (MPa)	Exact	−777.928	−1083.479	−1939.319	−2839.835	−4276.356	−4133.323	−3243.273	−2500.214	−1500.001	−500.000
	Present	−777.818	−1082.723	−1938.149	−2839.058	−4277.204	−4133.700	−3243.235	−2500.215	−1500.001	−500.000
σ_{22} (MPa)	Exact	−1153.572	−2116.829	−3608.249	−4532.583	−5068.363	−4046.915	−3247.699	−2499.997	−1500.001	−500.000
	Present	−1153.736	−2116.896	−3608.714	−4533.668	−5069.867	−4046.717	−3247.720	−2499.994	−1500.001	−500.000

Table 2
Comparison of the stress results obtained by the exact solution and the present solution in loading case 2

S (cm)		1000	990	970	950	900	800	650	500	300	100
σ_{11} (MPa)	Exact	-1.236×10^{-1}	-1.346×10^{-1}	-1.473×10^{-1}	-1.513×10^{-1}	-1.473×10^{-1}	-1.521×10^{-1}	-1.885×10^{-1}	-2.450×10^{-1}	-4.084×10^{-1}	−1.225
	Present	-1.236×10^{-1}	-1.346×10^{-1}	-1.473×10^{-1}	-1.513×10^{-1}	-1.473×10^{-1}	-1.520×10^{-1}	-1.885×10^{-1}	-2.450×10^{-1}	-4.084×10^{-1}	−1.225
σ_{12} (MPa)	Exact	-1.236×10^{-1}	-1.143×10^{-1}	-1.058×10^{-1}	-1.065×10^{-1}	-1.245×10^{-1}	-1.542×10^{-1}	-1.884×10^{-1}	-2.450×10^{-1}	-4.084×10^{-1}	−1.225
	Present	-1.236×10^{-1}	-1.143×10^{-1}	-1.057×10^{-1}	-1.064×10^{-1}	-1.245×10^{-1}	-1.542×10^{-1}	-1.884×10^{-1}	-2.450×10^{-1}	-4.084×10^{-1}	−1.225
σ_{21} (MPa)	Exact	-3.880×10^{-2}	-3.553×10^{-2}	-2.674×10^{-2}	-1.755×10^{-2}	-2.055×10^{-3}	1.225×10^{-3}	-6.181×10^{-5}	1.956×10^{-6}	1.103×10^{-8}	-1.364×10^{-12}
	Present	-3.880×10^{-2}	-3.554×10^{-2}	-2.676×10^{-2}	-1.756×10^{-2}	-2.047×10^{-3}	1.228×10^{-3}	-6.217×10^{-5}	1.976×10^{-6}	1.131×10^{-8}	-1.826×10^{-12}
σ_{22} (MPa)	Exact	-3.534×10^{-2}	-2.603×10^{-2}	-1.141×10^{-2}	-1.998×10^{-3}	5.222×10^{-3}	4.311×10^{-4}	-2.114×10^{-5}	-3.091×10^{-8}	5.251×10^{-9}	-5.397×10^{-12}
	Present	-3.534×10^{-2}	-2.603×10^{-2}	-1.140×10^{-2}	-1.988×10^{-3}	5.236×10^{-3}	4.293×10^{-4}	-2.095×10^{-5}	-5.227×10^{-8}	5.190×10^{-9}	-5.734×10^{-12}



(a) σ_{21}

(b) σ_{22}

Fig. 3. The relative error between the present solution and the approximate solution.

Introducing a complex variable

$$\tilde{\gamma}_1 = \gamma_1 + \mathrm{i}\frac{U}{Eh^2/\sqrt{12(1-\mu^2)}}. \tag{A.4}$$

Multiplying the first formula of the Eq. (1) by $\mathrm{i}/(Eh^2/\sqrt{12(1-\mu^2)})$ then adding to the second formula of Eq. (1) yields

$$L(\tilde{\gamma}_1) + \mathrm{i}\frac{\tilde{\gamma}_1}{h/\sqrt{12(1-\mu^2)}} = 0. \tag{A.5}$$

Letting

$$T = T_1 + T_2, \qquad M = M_1 + M_2, \tag{A.6}$$

and substituting Eqs. (A.2) and (A.3) into Eq. (A.6), we have

$$\begin{aligned} T &= -\left(\frac{\mathrm{d}U}{\mathrm{d}S} + \frac{\tan\alpha}{R_2}U\right), \\ M &= (1+\mu)D\left(\frac{\mathrm{d}\gamma_1}{\mathrm{d}S} + \frac{\tan\alpha}{R_2}\gamma_1\right). \end{aligned} \tag{A.7}$$

Complex stress resultants are defined as follows:

$$\begin{aligned} \tilde{T}_1 &= T_1 + \mathrm{i}\frac{Eh^2}{\sqrt{12(1-\mu^2)}}k_2, \\ \tilde{T}_2 &= T_2 + \mathrm{i}\frac{Eh^2}{\sqrt{12(1-\mu^2)}}k_1, \\ \tilde{T} &= \tilde{T}_1 + \tilde{T}_2 = T + \mathrm{i}\frac{\sqrt{12(1-\mu^2)}}{h}\frac{M}{1+\mu}. \end{aligned} \tag{A.8}$$

Substituting Eqs. (A.2), (A.3) and (A.7) into Eq. (A.8), the complex stress resultants expressed by complex argument $\tilde{\gamma}_1$ are obtained,

$$
\begin{aligned}
\tilde{T}_1 &= \mathrm{i}\frac{Eh^2}{\sqrt{12(1-\mu^2)}}\frac{\tan\alpha}{R_2}\tilde{\gamma}_1, \\
\tilde{T}_2 &= \mathrm{i}\frac{Eh^2}{\sqrt{12(1-\mu^2)}}\frac{\mathrm{d}\tilde{\gamma}_1}{\mathrm{d}S}, \\
\tilde{T} &= \mathrm{i}\frac{Eh^2}{\sqrt{12(1-\mu^2)}}\left(\frac{\mathrm{d}\tilde{\gamma}_1}{\mathrm{d}S}+\frac{\tan\alpha}{R_2}\tilde{\gamma}_1\right).
\end{aligned} \tag{A.9}
$$

Using the second formula of Eqs. (3), and (A.5) can be written as

$$
\frac{\mathrm{d}^2\tilde{\gamma}_1}{\mathrm{d}S^2}+\frac{1}{S}\frac{\mathrm{d}\tilde{\gamma}_1}{\mathrm{d}S}-\frac{1}{S^2}\tilde{\gamma}_1+\mathrm{i}\frac{1}{S\tan\alpha}\frac{1}{h/\sqrt{12(1-\mu^2)}}\tilde{\gamma}_1=0. \tag{A.10}
$$

In order to eliminate the term with first-order derivative with respect to S in Eq. (A.10), let us introduce new argument y which is defined as

$$
y=\tilde{\gamma}_1\exp\left(\frac{1}{2}\int\frac{1}{S}\mathrm{d}S\right)=\sqrt{S}\tilde{\gamma}_1. \tag{A.11}
$$

The first-order and second-order derivatives of new argument y with respect to S

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}S} &= \sqrt{S}\frac{\mathrm{d}\tilde{\gamma}_1}{\mathrm{d}S}+\frac{1}{2\sqrt{S}}\tilde{\gamma}_1, \\
\frac{\mathrm{d}^2y}{\mathrm{d}S^2} &= \sqrt{S}\frac{\mathrm{d}^2\tilde{\gamma}_1}{\mathrm{d}S^2}+\frac{1}{\sqrt{S}}\frac{\mathrm{d}\tilde{\gamma}_1}{\mathrm{d}S}-\frac{1}{4\sqrt{S^3}}\tilde{\gamma}_1.
\end{aligned} \tag{A.12}
$$

Substituting Eqs. (A.11) and (A.12) into Eq. (A.10), we have

$$
\frac{\mathrm{d}^2y}{\mathrm{d}S^2}-\frac{3}{4}\frac{1}{S^2}y+\mathrm{i}\frac{1}{S\tan\alpha}\frac{1}{h/\sqrt{12(1-\mu^2)}}y=0. \tag{A.13}
$$

Now let us perform the magnitude order analysis to every term of Eq. (A.13). As far as the general shell of revolution is concerned, boundary effect will decay rapidly away from the boundary edge. From the point of view of the magnitude order, calculating the derivative with respect to S every time is equivalent to multiplying by $\beta=\left(\sqrt[4]{3(1-\mu^2)}/\sqrt{h|R_{20}|}\right)$, i.e., dividing by $\sqrt{h|R_{20}|}$. Since the effect of the solution is only confined to local region near the boundary edge, it is reasonable that the values of the geometrical parameter in the local region, such as S, R_2, are considered to equal nearly to the values at the boundary edge S_0, R_{20}. Therefore, for conical shells, $S=R_2/\tan\alpha$, the ratio of magnitude order for every term in Eq. (A.13) becomes

$$
\frac{y}{h|R_{20}|}:\frac{y}{|R_{20}|^2/\tan^2\alpha}:\frac{y}{|R_{20}|h}.
$$

Multiplying every term by $h|R_{20}|$ yields

$$
y:\frac{h\tan^2\alpha}{|R_{20}|}y:y.
$$

If the conical shell is not too flat, the value of $\tan\alpha$ will not be very large, then the order of magnitude of the three terms is

$$
1:\frac{h}{|R_{20}|}:1.
$$

If we neglect the second term in Eq. (A.13), the error magnitude order will be $h/|R_{20}|$. Therefore, Eq. (A.13) can be simplified to

$$
\frac{\mathrm{d}^2y}{\mathrm{d}S^2}+\mathrm{i}\frac{1}{S\tan\alpha}\frac{1}{h/\sqrt{12(1-\mu^2)}}y=0. \tag{A.14}
$$

The proceeding equation can be written as

$$\frac{d^2y}{dS^2} - \tilde{F}^2 y = 0, \tag{A.15}$$

where

$$\tilde{F}^2 = -i\frac{1}{S\tan\alpha}\frac{1}{h/\sqrt{12(1-\mu^2)}}. \tag{A.16}$$

Solving Eq. (A.16), we have

$$\tilde{F} = \pm\frac{1-i}{\sqrt{2}}\frac{1}{\sqrt{cS\tan\alpha}}, \tag{A.17}$$

where

$$c = \frac{h}{\sqrt{12(1-\mu^2)}}. \tag{A.18}$$

Eq. (A.15) is a differential equation with variable coefficient. In order to transform Eq. (A.15) to the differential equation with constant coefficients we have to introduce the following argument transformations:

$$\begin{aligned}
&\xi = \frac{1}{\sqrt{2}}\int_{s_0}^{s}\frac{dS}{\sqrt{cS\tan\alpha}} = \frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{h\tan\alpha}}\left(2\sqrt{S} - 2\sqrt{S_0}\right),\\
&y_1 = by,\\
&b = \frac{1}{\sqrt[4]{cS\tan\alpha}} = \left(\frac{\tilde{F}}{\zeta}\right)^{1/2},\\
&\zeta = \pm\frac{1-i}{\sqrt{2}}.
\end{aligned} \tag{A.19}$$

Using Eq. (A.19), we can calculate the first-order and the second-order derivatives of ξ with respect to S and the first-order and the second-order derivatives of b with respect to ξ.

$$\begin{aligned}
&\frac{d\xi}{dS} = \frac{b^2}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{\tilde{F}}{\zeta},\\
&\frac{db}{d\xi} = \frac{1}{\sqrt{2}}\zeta^{1/2}\tilde{F}^{-3/2}\frac{d\tilde{F}}{dS},\\
&\frac{d^2b}{d\xi^2} = \zeta^{3/2}\left[\tilde{F}^{-5/2}\frac{d^2\tilde{F}}{dS^2} - \frac{3}{2}\tilde{F}^{-7/2}\left(\frac{d\tilde{F}}{dS}\right)^2\right].
\end{aligned} \tag{A.20}$$

From the second and the third formula of Eq. (A.19), we have

$$y = \frac{y_1}{b} = \left(\frac{\tilde{F}}{\zeta}\right)^{-1/2} y_1. \tag{A.21}$$

The derivatives of y with respect to S are

$$\begin{aligned}
&\frac{dy}{dS} = \frac{1}{\sqrt{2}}\left(b\frac{dy_1}{d\xi} - \frac{db}{d\xi}y_1\right),\\
&\frac{d^2y}{dS^2} = \frac{1}{2}\frac{\tilde{F}}{\zeta}\left[\left(\frac{\tilde{F}}{\zeta}\right)^{1/2}\frac{d^2y_1}{d\xi^2} - \frac{d^2b}{d\xi^2}y_1\right].
\end{aligned} \tag{A.22}$$

By means of the third formula of Eq. (A.20), the second equation of Eq. (A.22) can be written as

$$\frac{d^2y}{dS^2} = \frac{1}{2}\frac{\tilde{F}}{\zeta}\left\{\left(\frac{\tilde{F}}{\zeta}\right)^{1/2}\frac{d^2y_1}{d\xi^2} - \zeta^{3/2}\left[\tilde{F}^{-5/2}\frac{d^2\tilde{F}}{dS^2} - \frac{3}{2}\tilde{F}^{-7/2}\left(\frac{d\tilde{F}}{dS}\right)^2\right]y\right\}. \tag{A.23}$$

Substituting Eqs. (A.21) and (A.23) into Eq. (A.15), we obtain

$$\frac{d^2 y_1}{d\xi^2} - (1-i)^2 \left\{ 1 + \underline{\frac{1}{2\tilde{F}^2}\left[\frac{1}{\tilde{F}}\frac{d^2\tilde{F}}{dS^2} - \frac{3}{2}\frac{1}{\tilde{F}^2}\left(\frac{d\tilde{F}}{dS}\right)^2\right]} \right\} y_1 = 0. \tag{A.24}$$

According to Eqs. (A.17) and (A.18), we know that the argument $\tilde{F}$ only depends on the configuration size and material properties of the conical shell, therefore, the varying length of $\tilde{F}$ should be $|R_{20}|$. Based on the reason explained above the magnitude orders of the derivative of $\tilde{F}$ with respect to S are

$$\frac{1}{\tilde{F}}\frac{d^2\tilde{F}}{dS^2} \sim \frac{1}{\tilde{F}}\frac{d}{dS}\left(\frac{\tilde{F}}{|R_{20}|}\right) \sim \frac{1}{|R_{20}|^2},$$
$$\frac{1}{\tilde{F}^2}\left(\frac{d\tilde{F}}{dS}\right)^2 \sim \frac{1}{\tilde{F}^2}\left(\frac{\tilde{F}}{|R_{20}|}\right)^2 \sim \frac{1}{|R_{20}|^2},$$
$$\frac{1}{2\tilde{F}^2}\left[\frac{1}{\tilde{F}}\frac{d^2\tilde{F}}{dS^2} - \frac{3}{2}\frac{1}{\tilde{F}^2}\left(\frac{d\tilde{F}}{dS}\right)^2\right] \sim \frac{1}{2\tilde{F}^2}\frac{1}{|R_{20}|^2} \sim \frac{h}{|R_{20}|},$$

where the mark $\sim$ expresses magnitude order equivalent. According to the magnitude order analysis given above the underlined term in Eq. (A.24) can be neglected. The error magnitude order will be h/R_2 after neglecting the underlined term. Thus, Eq. (A.24) is simplified to

$$\frac{d^2 y_1}{d\xi^2} - (1-i)^2 y_1 = 0. \tag{A.25}$$

Eq. (A.25) is the differential equation with complex constant and contains only the second-order derivative term and the zero-order derivative term with respect to ξ. The solution to this differential equation is

$$y_1 = D_1' e^{-(1-i)\xi} + D_2' e^{(1-i)\xi}, \tag{A.26}$$

where D_1' and D_2' are unknown complex constants. By using Eq. (A.19), y can be obtained as

$$y = \sqrt[4]{S}\left[D_1'' e^{-(1-i)\xi} + D_2'' e^{(1-i)\xi}\right], \tag{A.27}$$

where D_1'' and D_2'' are unknown complex constants. The relations between D_1', D_2' and D_1'', D_2'' are

$$\begin{aligned} D_1'' &= \sqrt[4]{c\tan\alpha}\, D_1', \\ D_2'' &= \sqrt[4]{c\tan\alpha}\, D_2'. \end{aligned} \tag{A.28}$$

Using Eq. (A.11), we can obtain the complex argument $\tilde{\gamma}_1$,

$$\tilde{\gamma}_1 = \frac{1}{\sqrt[4]{S}}\left[D_1'' e^{-(1-i)\xi} + D_2'' e^{(1-i)\xi}\right]. \tag{A.29}$$

For simplicity, let

$$\begin{aligned} D_1'' &= \frac{2\lambda^2}{Eh} c_1 e^{i\zeta_1} \sqrt[4]{R_{20}}, \\ D_2'' &= \frac{2\lambda^2}{Eh} c_2 e^{i\zeta_2} \sqrt[4]{R_{20}}, \end{aligned} \tag{A.30}$$

where

$$\lambda = \sqrt[4]{3(1-\mu^2)}\sqrt{\frac{R_{20}}{h}}. \tag{A.31}$$

Substituting Eq. (A.30) into Eq. (A.29), we have

$$\tilde{\gamma}_1 = \frac{2\lambda^2}{Eh}\sqrt[4]{\frac{R_{20}}{S}}\left[c_1 e^{i\zeta_1} e^{-(1-i)\xi} + c_2 e^{i\zeta_2} e^{(1-i)\xi}\right]. \tag{A.32}$$

Now, we substitute Eq. (A.32) into Eq. (A.9) and then have

$$\begin{aligned}
\tilde{T}_1 &= \mathrm{i}\frac{Eh^2}{\sqrt{12(1-\mu^2)}}\frac{\tan\alpha}{R_2}\tilde{\gamma}_1, \\
&= \mathrm{i}\tan\alpha\frac{\sqrt[4]{R_{20}^5}}{R_2\sqrt[4]{S}}\left[c_1 \mathrm{e}^{\mathrm{i}\zeta_1}\mathrm{e}^{-(1-\mathrm{i})\xi} + c_2 \mathrm{e}^{\mathrm{i}\zeta_2}\mathrm{e}^{-(1-\mathrm{i})\xi}\right], \\
\tilde{T}_2 &= \mathrm{i}\frac{Eh^2}{\sqrt{12(1-\mu^2)}}\frac{\mathrm{d}\tilde{\gamma}_1}{\mathrm{d}S}, \\
&= \frac{\sqrt[4]{R_{20}^5}}{\sqrt[4]{S}}\left\{\left[-\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}} - \mathrm{i}\left(\frac{1}{4S} + \frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\right)\right] c_1 \mathrm{e}^{\mathrm{i}\zeta_1}\mathrm{e}^{-(1-\mathrm{i})\xi}\right. \\
&\quad \left. + \left[\frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}} - \mathrm{i}\left(\frac{1}{4S} - \frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{Sh\tan\alpha}}\right)\right] c_2 \mathrm{e}^{\mathrm{i}\zeta_2}\mathrm{e}^{-(1-\mathrm{i})\xi}\right\}.
\end{aligned} \tag{A.33}$$

Selecting the real part and the imaginary part of Eqs. (A.32) and (A.33), respectively, and using Eqs. (A.1)–(A.4) and (A.8), the stress resultants, bending moments and deformations can be calculated which are given in Section 4, Eqs. (30)–(35).

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